# Scattering Amplitudes, Riemann Spheres & Twistor Strings

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based on works with Freddy Cachazo & Ellis Yuan **(2013-15)**

with Yong Zhang **to appear**

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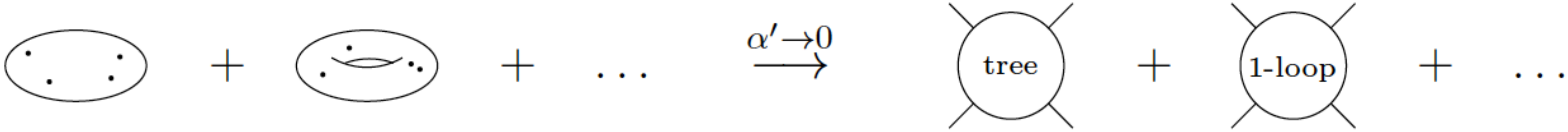
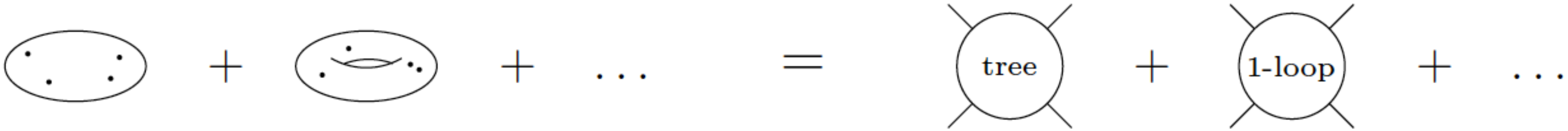
# Worldsheet picture for QFTs?

- Perturbative S-matrix in (massless) QFT as $\alpha' \to 0$ limit of usual string theory




- This talk: worldsheet formulas with localized modular integrals (no $\alpha'$) !




taken from Monteiro's talk

- A ``stringy" way of computing QFT amplitudes; dual to usual Feynman diagrams

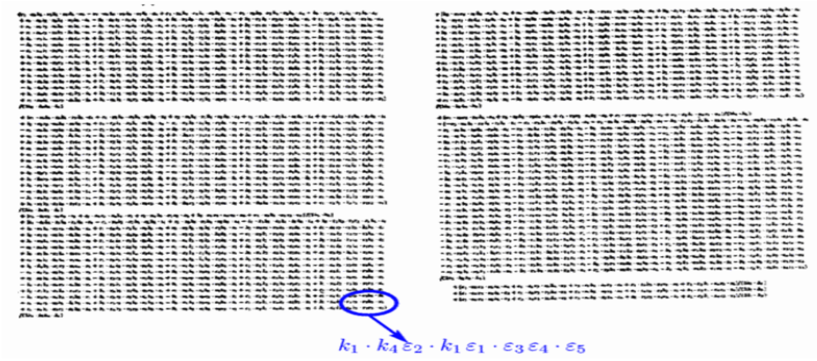
$\to$ suggest possible worldsheet description of QFTs with massless particles?

# Motivations: simplicity of S-matrix

- FDs are complicated: many diagrams, many many terms, gauge (non-)invariance

$n$-gluon scattering (tree)

| $n$ | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|
| # diagrams | 4 | 25 | 220 | 2485 | 34300 | 559405 | 10525900 |




- On-shell amplitudes in gauge theories & gravity: unexpected simplicity & structures obscured by off-shell redundancies in FDs, e.g. MHV gluon amplitudes [Parke, Taylor, 86]

$$M_n(i^-, j^-) = \frac{\langle i\, j\rangle^4}{\langle 1\, 2\rangle\, \langle 2\, 3\rangle \cdots \langle n\, 1\rangle},$$

$$k^\mu = (\sigma^\mu)_{\alpha,\dot\alpha} \lambda^\alpha \tilde\lambda^{\dot\alpha}, \quad \epsilon^\mu = \ldots$$

$$\langle a\, b\rangle := \varepsilon_{\alpha,\beta}\, \lambda_a^\alpha\, \lambda_b^\beta, \quad [a\, b] := \varepsilon_{\dot\alpha,\dot\beta}\, \tilde\lambda_a^{\dot\alpha}\, \tilde\lambda_b^{\dot\beta}$$

[Xu, Zhang, Chang, 84…]

- The beginning of 30 years progress on computing & understanding amplitudes

# Motivations: twistor string theory

- Witten's twistor string theory → worldsheet model for D=4, N=4 SYM

tree amps= string correlator w. $\mathbb{CP}^1 \to \mathbb{CP}^{3|4}$, or (super-)twistor space [Witten, 03]

- Key observation: [Nair, 88] Parke-Taylor MHV amps = correlator on $\mathbb{CP}^1$

$$\lambda_i^\alpha \sim (z_i, 1)\,, \qquad PT_n := \frac{1}{(z_1-z_2)\,(z_2-z_3)\cdots(z_n-z_1)}\,. \qquad j_A(z)j_B(z') = \frac{f_{AB}^C\, j_C}{z-z'} + \text{double poles} + \ldots$$

- A closed formula for all tree amplitudes (gluons etc.) in N=4 SYM [Roiban, Spradlin, Volovich, 04]

$N^{(k-2)}MHV$ amplitude is the image of $PT_n$ under polynomial map of deg. (k-1)

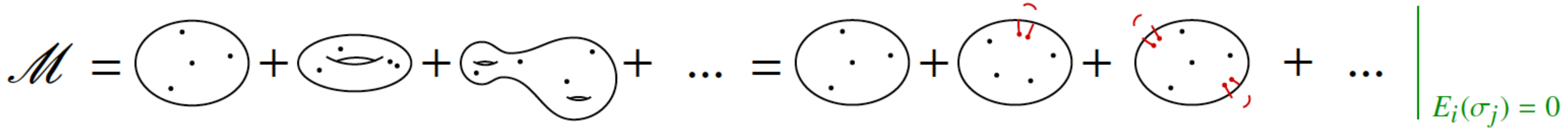
- Inspired CSW & BCFW, progress on unitarity method, Grassmannian, etc. etc.

# Cachazo-He-Yuan formulation

- Go beyond Witten's twistor string theory: no SUSY? any spacetime dim? more general theories: gauge theory, gravity, effective field theories? loop level???

- CHY formulation: scattering of massless particles in any dimension [CHY 2013]

  - *compact formulas* for amplitudes of gluons, gravitons, fermions, scalars, etc.
  - *manifest* gauge (diff) invariance, double-copy relations, soft/collinear behavior, etc.
  - *worldsheet picture:* "ambitwsitor"/"chiral" strings [Mason, Skinner; Adamo et al; Berkovits; Siegel,…]

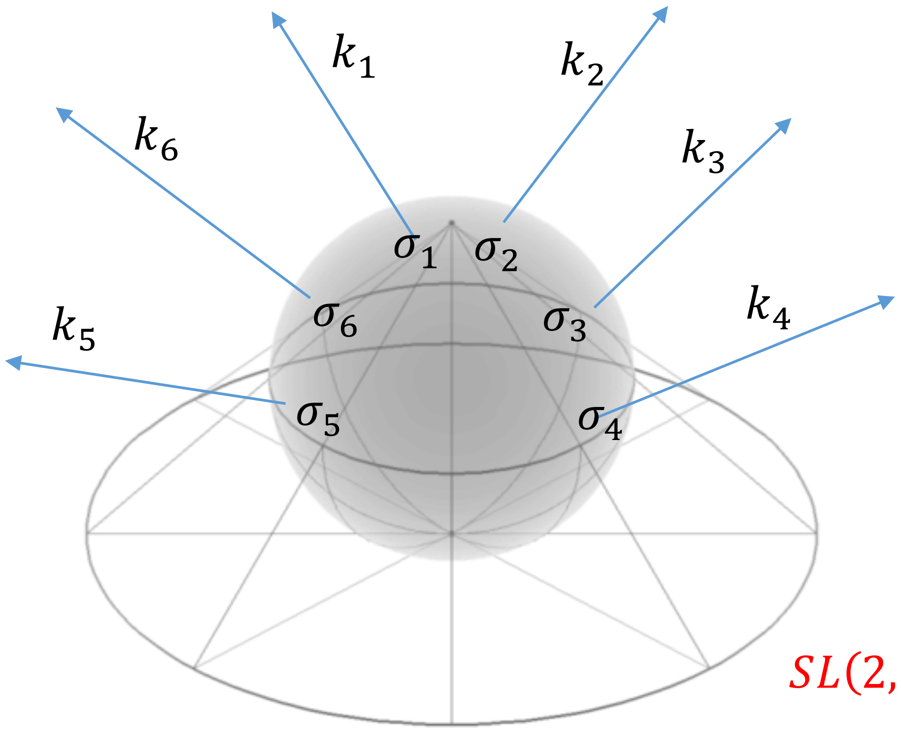



taken from Geyer's talk

# Scattering equations

$$E_a := \sum_{b=1, b\neq a}^{n} \frac{k_a \cdot k_b}{\sigma_a - \sigma_b} = 0\,, \qquad a = 1, 2, \ldots, n$$

[CHY 2013] [appeared in Farlie, Roberts, 72; Gross, Mende, 88;…]




$SL(2,\mathbb{C})$ symmetry

- universal, independent of theories; determine locations of punctures in terms of kinematics
- physical singularities ↔ boundary of moduli space for n-punctured Riemann spheres
- simplest "derivation": saddle point eqs in tensionless limit of string amps [Gross, Mende]

$$E_a = \frac{\partial[\sum_{i<j} s_{i,j} \ln(\sigma_{i,j})]}{\partial \sigma_a}$$

# CHY representation of tree amps

$$M_n = \underbrace{\int \frac{d^n\sigma}{\text{vol SL}(2,\mathbb{C})} \prod_a{}' \delta(E_a)}_{d\mu_n} \mathcal{I}(\{k,\epsilon,\sigma\}) = \sum_{\{\sigma\}\in\text{solns.}} \frac{\mathcal{I}(\{k,\epsilon,\sigma\})}{J(\{\sigma\})}$$

- S-matrix = localized integral = sum over solutions, of certain CHY integrand

  n-3 integrals/delta functions, (n-3)! solutions [CHY; Dolan, Goddard] → (n-3)! building blocks

- Picture: massless-particle scattering naturally via n-punctured Riemann spheres. Feynman diagrams and Lagrangians become emergent!

- Question: how to find "dynamic part", i.e. CHY integrands for different QFTs?

# CHY formulas: GR, YM & $\phi^3$

- All tree amplitudes in gravity, Yang-Mills & bi-adjoint $\phi^3$ theories in any dim [CHY 13]

$$M_n^{h+B+\phi} = \int d\mu_n \, \mathrm{Pf}'\Psi(\epsilon)\, \mathrm{Pf}'\Psi(\epsilon') \longrightarrow M_n^{\mathrm{GR}} = \int d\mu_n \, \det{}'\,\Psi(\epsilon)$$

$$M_n^{\mathrm{YM}}[\pi] = \int d\mu_n \, \mathrm{PT}[\pi]\, \mathrm{Pf}'\,\Psi$$

$$m[\pi|\rho] = \int d\mu_n \, \mathrm{PT}[\pi]\, \mathrm{PT}[\rho] \qquad \mathcal{L}_{\phi^3} = -\frac{1}{2}(\partial\phi)^2 + \frac{\lambda}{3!} f^{I\,J\,K} f^{I'\,J'\,K'}\, \phi^{I\,I'}\phi^{J\,J'}\phi^{K\,K'}$$

- Two ingredients: Parke-Taylor factor (color) & a Pfaffian (polarization): $PT^{2-s} \times Pf^{s}$

$$\mathrm{PT}[\pi] := \frac{1}{(\sigma_{\pi(1)} - \sigma_{\pi(2)})\,(\sigma_{\pi(2)} - \sigma_{\pi(3)}) \cdots (\sigma_{\pi(n)} - \sigma_{\pi(1)})} \qquad \mathrm{Pf}'\Psi(\sigma, k, \epsilon)$$

# The Pfaffian

- The (reduced) Pfaffian of a $2n\times 2n$ skew matrix $\Psi$ (two null vectors!)

$$\mathrm{Pf}'\Psi := \frac{\mathrm{Pf}|\Psi|^{i,j}_{i,j}}{\sigma_{i,j}} \qquad A_{a,b} := \begin{cases} \frac{k_a\cdot k_b}{\sigma_{a,b}} & a \neq b \\ 0 & a = b \end{cases}, \quad B_{a,b} := \begin{cases} \frac{\epsilon_a\cdot\epsilon_b}{\sigma_{a,b}} & a \neq b \\ 0 & a = b \end{cases},$$

$$\Psi := \begin{pmatrix} A & -C^T \\ C & B \end{pmatrix}, \quad C_{a,b} := \begin{cases} \frac{\epsilon_a\cdot k_b}{\sigma_{a,b}} & a \neq b \\ -\sum_{c\neq a} C_{a,c} & a = b \end{cases}$$

- From open superstring correlator: $\mathrm{Pf}'\Psi \sim \langle V^{(0)}(\sigma_1)\ldots V^{(-1)}(\sigma_i)\ldots V^{(-1)}(\sigma_j)\ldots V^{(0)}(\sigma_n)\rangle$

- Permutation invariance, multi-linearity in polarizations, correct weight & mass dim.

  most important: gauge invariance on the support of scattering eqs!

# Gauge & diffeomorphism invariance

- Pf' (det') $\Psi$ as the simplest gauge (diffeo. resp.) invariant object: $\epsilon_a^\mu \sim \epsilon_a^\mu + \alpha k_a^\mu$

$$\left(\begin{array}{cc|cc} 0 & \cdots & \sum_{b=2}^{n} \frac{k_1\cdot k_b}{\sigma_{1,b}} & \cdots \\ \frac{k_2\cdot k_1}{\sigma_{2,1}} & \cdots & \frac{k_2\cdot k_1}{\sigma_{2,1}} & \cdots \\ \vdots & & \vdots & \\ \frac{k_n\cdot k_1}{\sigma_{2,1}} & \cdots & \frac{k_n\cdot k_1}{\sigma_{2,1}} & \cdots \\ -\sum_{b=2}^{n} \frac{k_1\cdot k_b}{\sigma_{1,b}} & \cdots & 0 & \cdots \\ \frac{\epsilon_2\cdot k_1}{\sigma_{2,1}} & \cdots & \frac{\epsilon_2\cdot k_1}{\sigma_{2,1}} & \cdots \\ \vdots & & \vdots & \\ \frac{\epsilon_n\cdot k_1}{\sigma_{2,1}} & \cdots & \frac{\epsilon_n\cdot k_1}{\sigma_{2,1}} & \cdots \end{array}\right)$$

- It vanishes by scattering equations
- (n-3)! gauge-inv. building blocks
- closed-string=(open-string)^2

# Double-copy relations "$GR\sim YM^2$"

- First double-copy relations from FT limit of Kawai-Lewellen-Tye relations:

$$M_n^{\text{closed}} = \sum_{\alpha,\beta} M_n^{\text{open}}[\alpha]\ \mathcal{S}^{\text{string}}[\alpha|\beta]\ M_n^{\text{open}}[\beta] \quad \Longrightarrow \quad M_n^{\text{GR}} = \sum_{\alpha,\beta} M_n^{\text{YM}}[\alpha]\ S[\alpha|\beta]\ M_n^{\text{YM}}[\beta]\,.$$

- More precisely $GR\sim YM^2/\phi^3$: naturally derived in CHY and found $\boldsymbol{S} = \boldsymbol{m}^{-1}$ !

$$M_n = \int d\mu_n\, \mathcal{I}_L\, \mathcal{I}_R \quad \Longrightarrow \quad M_n = \sum_{\alpha,\beta} M_L[\alpha]\ m^{-1}[\alpha|\beta]\ M_R[\beta], \quad \text{for } M_{L(R)} := \int d\mu_n\, \text{PT}\ \mathcal{I}_{L(R)}\,.$$

- A general way of seeing double-copy relations: splitting a CHY formula into two.

# More theories

- Generate CHY formulas of new theories from old ones, e.g. dim reduction

  GR → Einstein-Maxwell (EM), YM → YM-scalar (YMs), with Pfaffian factorizes:

$$M_{n\,\gamma}^{\mathrm{EM}} = \int d\mu_n\, \mathrm{Pf}'A\, \mathrm{Pf}X\ \mathrm{Pf}'\Psi\,, \quad M_{n\,s}^{\mathrm{YMs}} = \int d\mu_n\, \mathrm{Pf}'A\, \mathrm{Pf}X\ \mathrm{PT}\,; \quad X_{a\,b} = \frac{\delta^{I_a\, I_b}}{\sigma_{a,b}}(1-\delta_{a,b})\,.$$

- A new operation to add non-abelian interactions leads to "direct sum" of theories Formulas in Einstein ⊕ Yang-Mills, YM ⊕ bi-adjoint scalar theories [CHY 14][Cachazo et al 16]

- A new class: "exceptional" effective field theories (EEFT) with Goldstone scalars

  very special theories: amplitudes have enhanced "Adler's zero"! [Cheung et al 14] [CHY 14]

# More theories

- $M_n = \int d\mu_n \, (\text{Pf}'A)^2 \, \text{PT}$, adjoint scalars with two derivative coupling?

  U(N) NLSM (the chiral Lagrangian) $\mathcal{L} = \text{Tr}(\partial_\mu U^+ \partial^\mu U)$

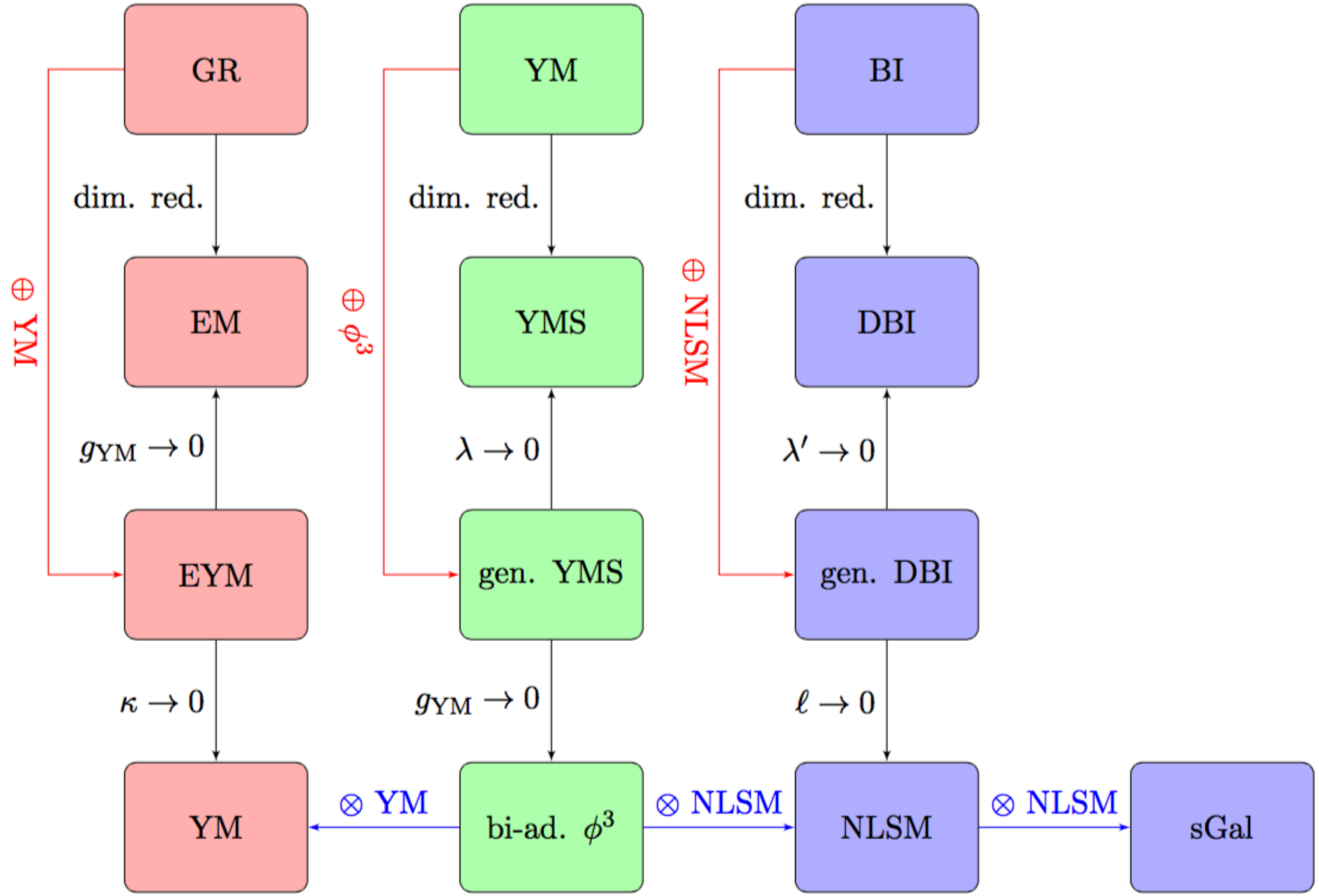
- $M_n = \int d\mu_n \, (\text{Pf}'A)^2 \, \text{Pf}'\Psi$, higher-derivative-coupled photons?

  Born-Infeld theory (BI) & DBI by dim reduction $\mathcal{L} = \sqrt{-\det(\eta_{\mu\nu} - \ell F_{\mu\nu} - \ell^2 \partial_\mu\phi\partial_\nu\phi)}$

- a special Galileon (single scalar with many derivatives) $M_n^{\text{sGal}} = \int d\mu_n \, (\text{Pf}'A)^4$

- double-copy relations: $BI \sim YM \otimes NLSM$, $DBI \sim YMs \otimes NLSM$, $sGal \sim NLSM^2$

# Roadmap of massless theories



# Soft theorems in CHY

- CHY makes manifest old & new soft theorems; connections to BMS etc. [Strominger,…]

$$M_n^{\text{gauge}} = (\mathcal{S}^{(0)}+\mathcal{S}^{(1)})\, M_{n-1}^{\text{gauge}} + O(\tau)\,, \quad M_n^{\text{GR}} = (\mathcal{S}^{(0)}+\mathcal{S}^{(1)}+\mathcal{S}^{(2)})M_{n-1}^{\text{GR}} + O(\tau^2)\,,$$

- EEFT's are the only scalar EFT's with vanishing soft behavior as O($\tau^p$) for p=1,2,3
  Non-linearly realized symmetry: coset from double-soft-scalar theorems [CHY 15]

$$M_n^{\text{NLSM}} = (S^{(0)}+S^{(1)})\, M_{n-2}^{\text{NLSM}} + O(\tau^2)\,, \quad M_n^{\text{DBI}} = (S^{(0)}+S^{(1)}+S^{(2)})M_{n-2}^{\text{DBI}} + O(\tau^4)\,,$$

(also for sGal). Striking similarities with gauge/gravity soft theorems. Why?

- 4d: manifest double soft theorems in N=4 SYM, N=8 SUGRA & DBI-Volkov-Akulov

  e.g. double fermions: non-linearly realized SUSY [Huang et al 14] $\leftrightarrow$ 16+16 [Bergshoeff et al 14]

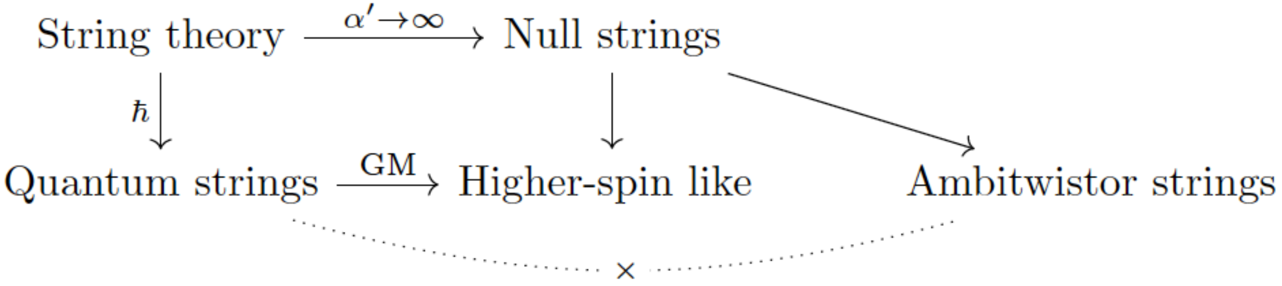
# Worldsheet origin

- Chiral string theory in "ambitwistor space"(=space of complex null geodesics) [Mason, Skinner 13]

$$S_{\text{bos.}} = \frac{1}{2\pi}\int \left(P^\mu\,\bar\partial X_\mu - \frac{1}{2}\,e\,P^2\right) \implies P^\mu(\sigma) = d\sigma \sum_{a=1}^{n} \frac{k_a^\mu}{\sigma - \sigma_a}$$

e enforces $P^2 = 0 \Leftrightarrow$ scattering equations. No $\alpha'$, only massless spectrum; coupled to matter systems & quantizations -> CHY formulas for gravity, YM, etc.

- Quantization of null strings; related to usual strings only in tensionless limit [Casali, Tourkine, 16]



# Loops from trees

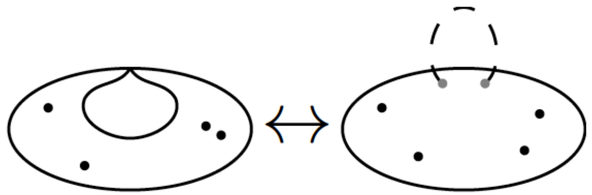
- Originally from ambitwistor strings on a torus (g=1) for SUGRA [Adamo-Casali-Skinner, 13]

$$M_{\rm SG}^{(1)} = \int d^D\ell \, d\tau \, \delta(P^2) \prod_{a=2}^{n} dz_a \delta(k_a \cdot P) \left( \sum_{\rm spin.str.} \mathcal{I}(z|\tau) \right) , \qquad P^\mu = dz \left( \ell^\mu + \sum_{a=1}^{n} k_a^\mu \frac{\theta_1'(z-z_a)}{\theta_1(z-z_a)} \right)$$

- Hard to evaluate; residue theorem localizes on $q = e^{2\pi i \tau} = 0$ [Geyer-Mason-Monteiro-Tourkine, 15]

  torus → nodal Riemann spheres: eqs & formula similar to trees!

$$P^\mu = d\sigma \left( \frac{\ell^\mu}{\sigma - 0} - \frac{\ell^\mu}{\sigma - \infty} + \sum_{a=1}^{n} \frac{k_a^\mu}{\sigma - \sigma_a} \right) , \qquad \mathcal{E}_a = k_a \cdot P(\sigma_a)$$



$$\tau \to i\infty : \quad \mathcal{I}_n^{\rm SUGRA} = \mathcal{P}_n(\epsilon) \, \mathcal{P}_n(\tilde{\epsilon}) , \quad \mathcal{P}_n := \left( {\rm Pf} M_3 \, |_{q^{1/2}} - 8 {\rm Pf} M_3 \, |_{q^0} \right) - 8 {\rm Pf} M_2 \, |_{q^0} \, .$$

- Progress at two loops [Geyer et al 16] ; ambitwistor strings for SYM etc., not clear!

# Loops from trees

- Feynman's tree theorem → loop amps from (generally divergent) forward limits of trees [c.f. Caron-Huot ,10]; need off-shell momenta ($l^2 \neq 0$) & regularizations, e.g.

$$M_n^{1-\text{loop}} \sim \int \frac{d^D \ell}{\ell^2} \sum_{l_+=l_-,\epsilon_+=(\epsilon_-)^*} M_{n+2}^{\text{tree}}( \{(k_i;0)\}, \ \pm(\ell,|\ell|) ) \,,$$

- Both resolved in CHY rep: loop-level eqs & formulas on a sphere [Geyer et al 15] [CHY 15]

$$M_n^{(1)} = \int d^D\ell \frac{1}{\ell^2} \int d\mu_n^{(1)} \, \mathcal{I}_n(\{\sigma,k,\epsilon\};\ell) \,, \quad \mathcal{E}_a = \sum_{b\neq a} \frac{k_a \cdot k_b}{\sigma_a - \sigma_b} + \frac{k_a \cdot \ell}{\sigma_a} \,, \quad \text{for } a = 1,\ldots,n \,.$$

- Seem wrong? → a new rep of loop integrands (diff. integrate to zero!) [Baadsgaard et al 15]

$$\frac{1}{\prod_i D_i} = \sum_i \frac{1}{D_i \ \prod_{j\neq i}(D_j - D_i)} \,; \ \text{shift } D_i \to \ell^2 \quad \Longrightarrow \quad \frac{1}{\ell^2} \sum_i \frac{1}{\prod_j \left(2\ell \cdot K_j^{(i)} + (K_j^{(i)})^2\right)}$$

# Loops from trees

- Suming over colors/polarizations gives 1-loop “PT” & “Pfaffians” [Geyer et al 15][HY, CHY 15]

$$\mathrm{PT}_n^{(1)}[1,2,\ldots,n] := \sum_{i=1}^{n} \mathrm{PT}_{n+2}[1,\ldots,i,+,-,i{+}1,\ldots,n]\,.$$

$$\mathrm{Pf}_{\mathbf{s}}^{(1)} = \frac{1}{\sigma_{+,-}^2}\mathrm{Pf}\Psi_n(\ell), \quad \mathrm{Pf}_{\mathbf{g}}^{(1)} = \sum_{\epsilon_+=(\epsilon_-)^*} \mathrm{Pf}'\Psi_{n+2}(\ell), \quad \mathrm{Pf}_{\mathbf{f}}^{(1)} = \ldots$$

- One-loop formula for $\phi^3$, YM & GR; also susy-theories by adding fermions [Geyer et al 15]

$$\mathcal{I}_n^{\phi^3} = (\mathrm{PT}_n^{(1)})^2, \quad \mathcal{I}_n^{\mathrm{YM}} = \mathrm{PT}_n^{(1)}\,\mathrm{Pf}_{\mathbf{g}}^{(1)}, \quad \mathcal{I}_n^{\mathrm{GR}} = (\mathrm{Pf}_{\mathbf{g}}^{(1)})^2 - c_d(\mathrm{Pf}_{\mathbf{f}}^{(1)})^2\,,$$

$$\mathcal{I}_n^{\mathrm{SYM}} = \mathrm{PT}_n^{(1)}\,(\mathrm{Pf}_{\mathbf{g}}^{(1)} - c_d\mathrm{Pf}_{\mathbf{f}}^{(1)})\,, \quad \mathcal{I}_n^{\mathrm{SUGRA}} = (\mathrm{Pf}_{\mathbf{g}}^{(1)} - c_d\mathrm{Pf}_{\mathbf{f}}^{(1)})^2\,.$$

- Gauge invariance, soft theorems, double-copy relations etc. manifest @ loop level;

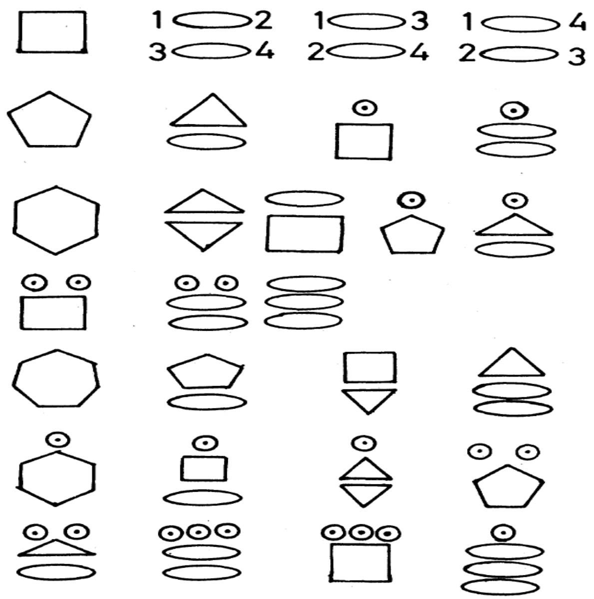
# New formulas for SYM & SUGRA

- Puzzle: SUSY formulas more complicated due to spin structures? (familiar in RNS) → Perform spin sum in the same way as that for g=1 superstring correlator [Tsuchiya, 88]

$$\mathcal{P}_n \sim \sum_{\text{cyc.}} \text{tr}(f\,f\ldots)\,\text{tr}(f\,f\ldots)\,\mathcal{G}_n(\{z\}|\tau) + 1\text{−cycs. terms}\,, \quad \mathcal{G}: \text{ferm. corr.}$$

$$\mathcal{G}_i(x_0, x_1, \ldots, x_{i-1}) = \sum_{\nu=1,2,3} (-)^\nu \left(\frac{\theta_{\nu+1}(0|\tau)}{\dot\theta_1(0|\tau)}\right)^4 \prod_{j=0}^{i-1} \frac{\dot\theta_1(0|\tau)\theta_{\nu+1}(x_j|\tau)}{\theta_{\nu+1}(0|\tau)\theta_1(x_j|\tau)}\,,$$

$$\Rightarrow \mathcal{G}_i = 0\,, \quad \text{for } i < 4\,; \quad \mathcal{G}_4 = 1\,; \quad \mathcal{G}_5 = \sum_{j=0}^{4} \frac{\partial}{\partial x_j} \ln\theta_1(x_j)\,, \ldots$$




- Key: much simpler fermionic part in field theory: full $\sigma$-dep. through $G_{i,j} \equiv \frac{\sigma_i+\sigma_j}{\sigma_i-\sigma_j}$

# New formulas for SYM & SUGRA

- Reorganize the spin-summed result in terms of powers in G's and loop momentum $l$

$$\mathcal{P}_n = \text{Soft}(\ell) \otimes \mathcal{P}_{n-1} + \sum_{m=0/1}^{n-4} \sum_{i_1,j_1;\ldots;i_m,j_m} G_{i_1,j_1} \cdots G_{i_m,j_m} T_n^{i_1,j_1;\ldots;i_m,j_m}(\epsilon, k)$$

$$\text{Soft}_i = \epsilon_i \cdot \ell + \frac{1}{2} \sum_{j \neq i} \epsilon_i \cdot k_j \, G_{i,j} \,, \quad T_n \sim \text{tr}(f_1 f_2 \cdots) \, \text{tr}(f_3 f_4 \cdots) \cdots \text{, computed up to 8pts}$$

- Gauge-invariant & soft manifest decomposition; recursive structures for the T's:

$$\mathcal{P}_4 = T_4 = \text{tr}(f_1 f_2 f_3 f_4) + \text{perm.} - (\text{tr}(f_1 f_2) \, \text{tr}(f_3 f_4) + \text{perm.}) \equiv t_8 f^4$$

$$\mathcal{P}_5 = \text{Soft} \otimes \mathcal{P}_4 + \sum_{i,j=1}^{5} G_{i,j} T_5^{i,j} \,, \quad \text{e.g. } T_5^{1,2} = \text{tr}(f_1 f_2 f_3 f_4 f_5) - \frac{1}{2} \text{tr}(f_1 f_2 f_3) \, \text{tr}(f_4 f_5) + \text{perm}[3,4,5]$$

# New formulas for SYM & SUGRA

- New representation of 1-loop SYM/SUGRA amps: gauge invariance vs. locality

$$M_n^{\rm SYM} = \int \frac{d^D\ell}{\ell^2} \int \prod_{i=2}^{n} \frac{d\sigma_i}{\sigma_i}\, \delta(\mathcal{E}_i)\, \mathrm{PT}_n^{(1)}\, \mathcal{P}_n\,, \quad M_n^{\rm SUGRA} = \int \frac{d^D\ell}{\ell^2} \int \prod_{i=2}^{n} \frac{d\sigma_i}{\sigma_i^2}\, \delta(\mathcal{E}_i)\, \mathcal{P}_n(\epsilon)\, \mathcal{P}_n(\tilde\epsilon)\,.$$

closely related to/shed new lights on field-theory limit of string amps [Mafra, Schlotterer 14-16]

- Makes SUSY manifest: up to $l^{n-4}$ or $G^{n-4}$ structures, enormous simplifications in 4d!

$$4d \text{ with helicities } 1^-,\ldots k^-;(k{+}1)^+,\ldots,n^+: \quad \mathcal{P}_n = H_k^{(1)}\,\tilde H_{n-k}^{(1)}$$

$$H_0^{(1)} = H_1^{(1)} = 0\,, \quad H_2^{(1)} = \langle 1\,2\rangle^2\,, \quad H_3^{(1)} = (G_{1,2}{+}G_{2,3}{+}G_{3,1})\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 3\,1\rangle{+}S_1^-\,\langle 2\,3\rangle^2{+}\text{cyc.}\;\ldots$$

- Towards "twistor-string" formulas for loop integrands in N=4 SYM & N=8 SUGRA

# Back to four dimensions

- CHY rep. simplifies in 4d → old & new “twistor-string” formulas [RSV 04; Cachazo, Skinner 12; …]

- Key: scattering eqs split into RSV-Witten eqs for sectors, k=2,3…,n-2 [CHY 13; SH, Z. Liu, J. Wu 16].

  *A prior* nothing to do with helicities, any amp decomposes into n-3 sectors:

$$M_n = \sum_{k=2}^{n-2} \sum_{\text{soln. } k} \frac{I_n^{\text{CHY}}}{J_n^{\text{CHY}}} := \sum_{k=2}^{n-2} M_{n,k}\,, \quad \Rightarrow \quad M_{n,k} = \int d\mu_{n,k}^{\text{4d}}\; I_{n,k}^{\text{4d}}\,.$$

- 4d eqs vs. Grassmannian: with C($\sigma$) ∈ G(k,n), $C \cdot \tilde{\lambda}=0$, $C^{\perp} \cdot \lambda=0$. [Arkani-Hamed et al 08]

  fix GL(k): $\tilde{\lambda}_I^{\dot{\alpha}} - \sum_{i=k+1}^{n} \frac{\tilde{\lambda}_i^{\dot{\alpha}}}{(I\,i)} = 0\,,\ I = 1,\ldots,k; \quad \lambda_i^{\alpha} - \sum_{I=1}^{k} \frac{\lambda_I^{\alpha}}{(i\,I)} = 0\,,\ i = k+1,\ldots,n\,.$ [Geyer et al 14]

# Back to four dimensions

- YM & GR with k neg. hel., Pf' $\Psi$=0 for any soln. sector $k' \neq k$! [SH, Zhang 16]

  for EEFT's: Pf' A=0 for $k' \neq \frac{n}{2}$; 4d formulas simplify a lot & natural to have SUSY

- 4d integrands in SYM, SUGRA & DBIVA (w. susy-measures) [Geyer et al 14; SH, Liu, Wu 16; …]

$$\mathcal{I}_{n,k}^{\text{SYM,4d}} = \text{PT}_n\,, \quad \mathcal{I}_{n,k}^{\text{SUGRA,4d}} = \det{}' H_k \det{}' \tilde{H}_{n-k}\,, \quad \mathcal{I}_{n,\frac{n}{2}}^{\text{DBIVA,4d}} = \det{}' A_n\,.$$

- Easy to include fermions in 4d, e.g. gluon-quark amps in massless QCD [SH, Zhang 16]

$$M_{n,k}(g;q\bar{q}) = \int d\mu_{n,k}\, \text{PT}_n\, \mathcal{J}_{\text{ferm}}\,,$$

Jac. depends on helicities & flavors of quarks only!

follow from gluon-gluino ones in SYM → new rep for all QCD tree amplitudes

# Summary & outlook

- New picture: massless particles scattering via punctured Riemann spheres.
  suggest a weak-weak duality: QFT S-matrix vs. string/worldsheet correlators?
- Complimentary to FD's: (n-3)! building blocks with symmetries manifest
- Web of theories connected by $\oplus$ (interaction) & $\otimes$ (double-copy). Scope of QFTs?
- Loops from nodal Riemann spheres, with new structures for SYM & SUGRA
- A bridge between S-matrix program in QFTs and string perturbation theory

# Thank You!